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Motions in a Bose condensate: IV. Axisymmetric solitary waves

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Abstract. Axisymmetric disturbances that preserve their form as they move through a Bose condensate are obtained numerically by the solution of the appropriate nonlinear Schrödinger equation. A continuous family is obtained that, in the momentum (p)–energy (E) plane, consists of two branches meeting at a cusp of minimum momentum around 0.140 $\rho\kappa^3/c^2$ and minimum energy about 0.145 $\rho\kappa^3/c$, where ρ is density, c is the speed of sound and κ is the quantum of circulation. For all larger p, there are two possible energy states. One (the lower branch) is (for large enough p) a vortex ring of circulation κ; as $p \to \infty$ its radius $\tilde{\omega} \sim (p/\pi\kappa)^{1/2}$ becomes infinite and its forward velocity tends to zero. The other (the upper branch) lacks vorticity and is a rarefaction sound pulse that becomes increasingly one dimensional as $p \to \infty$; its velocity approaches c for large p. The velocity of any member of the family is shown, both numerically and analytically, to be $\partial E/\partial p$, the derivative being taken along the family. At great distances, the disturbance in the condensate is pseudo-dipolar (dipolar in a stretched coordinate system); the strength of the pseudo-dipole moment is obtained numerically. Analogous calculations are presented for the corresponding two-dimensional problem. Again, a continuous sequence of solitary waves is obtained, but the momentum per unit length p and energy per unit length E have no minima. For small forward velocities, the wave consists of two widely separated parallel, oppositely directed line vortices. As the forward velocity increases the wave loses its vorticity and becomes a rarefaction pulse of ever increasing spatial extent but ever decreasing amplitude. As its velocity approaches c, both p and E tend to zero, and $E/p \to c$.

1. Introduction

This paper takes up questions left unanswered in the first of the three previous papers in this series, one devoted to the flow of an imperfect Bose gas (Roberts and Grant 1971, Grant 1971, Grant and Roberts 1974). In that paper the structure, velocity U, momentum (impulse) p, and energy $\mathscr{E}$ of a large vortex ring were determined asymptotically in the limit in which the radius $\tilde{\omega}$ of the ring is infinite. It was conjectured there (as it was later by Huggins 1971, Ichiyanagi 1979) that, as for the classical vortex ring (Norbury 1972, 1973), these vortices are extreme members of a sequence that probably extends to zero $\tilde{\omega}$. It was clear however that, once $\tilde{\omega}$ became comparable to the healing length (that determines the core radius of the large ring), it would be impossible to use asymptotic methods: numerical integration would be the only recourse.

In so far as a condensate is a qualitatively faithful model of superfluid helium, considerable interest attaches to determining the entire vortex sequence, for its members define possible states that can be excited in helium. The large vortex ring was, for instance, long ago experimentally detected by Rayfield and Reif (1964). There has also been the suggestion that such a vortex branch should merge continuously with the phonon–roton dispersion curve. The roton could then, as Onsager conjectured, be pictured as 'the ghost of a vanished vortex ring' (see Donnelly (1974) for the history of this idea).

Even those who feel that the condensate is too crude a model of helium II to be a reliable guide in these matters, will concede that the new solutions to the nonlinear Schrödinger equation which this paper presents are of interest when so few have been derived. Our axisymmetric solutions are, as far as we are aware, the only completely solitary wave solutions known, i.e. disturbances that are form preserving and vanish with distance in *all* directions from the centre of the wave.

The imperfect Bose condensate is governed by equations that were derived by Gross and by Ginzberg and Pitaevski. In a Hartree approximation, the single-particle wavefunction $\psi(\boldsymbol{x}, t)$ for the N bosons of mass M that fill the volume V obeys the nonlinear Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi + W_0\psi|\psi|^2 \tag{1.1}$$

where W_0 is the strength of the assumed δ-function repulsive potential between bosons. With E_v the average energy level per unit mass of a boson, we write

$$\psi = \exp(-\mathrm{i}ME_vt/\hbar)\Psi \tag{1.2}$$

so that by (1.1)

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi + W_0\Psi|\Psi|^2 - ME_v\Psi. \tag{1.3}$$

We cast this in dimensionless form by the transformation

$$\boldsymbol{x} \to \frac{\hbar}{(2M^2E_v)^{1/2}}\boldsymbol{x} \qquad t \to \frac{\hbar}{2ME_v}t \qquad \Psi \to \rho_\infty^{1/2}\Psi$$

where $\rho_\infty = ME_v/W_0$, giving

$$2\mathrm{i}\,\partial\Psi/\partial t = -\nabla^2\Psi - \Psi(1-|\Psi|^2). \tag{1.4}$$

The hydrodynamic interpretation of (1.4) will often be useful. This is obtained from the Madelung transformation

$$\Psi = R\,\mathrm{e}^{\mathrm{i}S} \tag{1.5}$$

where R and S are real. Substituting into (1.4), separating real and imaginary parts, and introducing fluid density ρ and fluid velocity $\boldsymbol{u}$ by

$$\rho = R^2 \qquad \boldsymbol{u} = \nabla S \tag{1.6), (1.7}$$

we recover the usual mass continuity equation,

$$\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{u}) = 0 \tag{1.8}$$

and a less usual Bernoulli equation involving a quantum potential

$$\partial S/\partial t + \tfrac{1}{2}\boldsymbol{u}^2 + \tfrac{1}{2}(\rho - 1) - \tfrac{1}{2}\rho^{-1/2}\nabla^2\rho^{1/2} = 0. \tag{1.9}$$

It is worth noting for future reference (§ 2) that the actual equation of motion, obtained by taking the gradient of (1.9), is

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial \Sigma_{ij}}{\partial x_j} \tag{1.10}$$

where the stress tensor Σ_{ij} is (Hills and Roberts 1977)

$$\Sigma_{ij}=[\tfrac{1}{4}(1-\rho^2)+\tfrac{1}{2}(\nabla\rho^{1/2})^2+\tfrac{1}{2}\rho^{1/2}\nabla^2\rho^{1/2}]\delta_{ij}-\frac{\partial\rho^{1/2}}{\partial x_i}\frac{\partial\rho^{1/2}}{\partial x_j}. \tag{1.11}$$

It is obvious from (1.10) that Σ_{ij} is arbitrary to an additive constant tensor, and for convenience we have taken this to be $\frac{1}{4}\delta_{ij}$ in order that Σ_{ij} vanishes at infinity where $\rho=1$. Other choices would not affect our results.

According to (1.7), the flow is irrotational and by (1.5) the velocity potential S is single valued up to an arbitrary multiple of 2π, it being always necessary that Ψ be single valued. Of particular interest to us are curves on which Ψ possesses a simple zero, and round which, therefore, S increases or decreases by 2π. These define vortex lines with a single unit of circulation, 2π, or $\kappa=h/M$ in dimensional units. Such curves can only be closed, or terminate on the walls of the container.

In what follows we seek solitary wave solutions of (1.4), i.e. solutions that satisfy two conditions. First, they are form-preserving: for each value of a dimensionless wave speed, U, we have

$$\Psi(x,y,z,t)=\Psi(x,y,\eta) \tag{1.12}$$

where $\eta=z-Ut$. Now

$$\frac{\partial}{\partial t}=-U\frac{\partial}{\partial\eta}\qquad\frac{\partial}{\partial z}=\frac{\partial}{\partial\eta} \tag{1.13}$$

and (1.4) for instance gives (at $t=0$, say, where $\eta=z$)

$$2\mathrm{i}U\frac{\partial\Psi}{\partial z}=\nabla^2\Psi+\Psi(1-|\Psi|^2). \tag{1.14}$$

Second, the disturbance associated with the wave must vanish at great distance, where the fluid is left undisturbed for our three-dimensional solutions

$$\Psi\to 1\qquad |\boldsymbol{x}|\to\infty. \tag{1.15}$$

(Note $\Psi=1$ obeys (1.4) and (1.15) and implies that $\rho=1$ and $\boldsymbol{u}=0$.) The significant point about (1.15) is that it applies for *all* directions of $\boldsymbol{x}$.

It is evident from (1.11) that the fluid is compressible. Indeed, 'in the bulk' (regions far from walls and vortices) where gradients of ρ are negligible, (1.9) or (1.10) show that the pressure P is (apart from a constant) proportional to ρ^2. In general, therefore, no stream function exists for $\boldsymbol{u}$. The solitary wave, however, is steady in the co-moving frame so that a stream function exists for $\rho\boldsymbol{u}$:

$$\boldsymbol{u}=\boldsymbol{U}+\rho^{-1}\nabla\times\boldsymbol{A}. \tag{1.16}$$

For the axisymmetric waves on which we shall concentrate, $\boldsymbol{A}=\Phi(s,z)\hat{\boldsymbol{\varphi}}/s$ so that

$$\boldsymbol{u}=U\hat{\boldsymbol{z}}+\frac{1}{\rho}\nabla\times\left(\frac{\Phi}{s}\hat{\boldsymbol{\varphi}}\right). \tag{1.17}$$

Here $(\hat{s}, \hat{\varphi}, \hat{z})$ are unit vectors in the direction of increasing cylindrical coordinates (s, φ, z). We will later present numerical solutions of (1.14) by means of density (ρ) and streamline (Φ) surfaces.

After defining quantities of interest in § 2, such as pseudo-dipole moment m, momentum (impulse) p, and energy $\mathscr{E}$ of the solitary waves, we present the results of numerical integrations in § 3. A continuous sequence of solitary waves is located. At one extreme lies the ring vortex of infinite radius, $\tilde{\omega}$, as defined by a single curve $(s = \tilde{\omega}, z = 0)$ of zero Ψ. As $\tilde{\omega}$ decreases so do m, p and $\mathscr{E}$, but U increases. Eventually $\tilde{\omega}$ becomes zero and all vorticity disappears, but the sequence continues as rarefaction pulses, U continues to increase, and p and $\mathscr{E}$ soon simultaneously reach a minimum and thereafter increase with U, becoming infinite together as U approaches the speed of sound. Thus, for each p greater than the minimum, there are two possible states, one on the 'lower' (vortex) branch and the other on the 'upper' (rarefaction pulse) branch. The relationship of the $p \to \infty$ solutions of the upper branch to the Tsuzuki (1971) soliton is discussed in appendix 1, where it is shown that $p \sim \frac{8}{3}\pi m$ in this limit. (As $p \to \infty$ on the lower branch, we recover the classical result for vortices, $p \sim 4\pi m$.)

The whole programme was repeated for two-dimensional solutions of (1.14), i.e. those for which $\Psi = \Psi\,(x, z - Ut)$ and (1.15) is replaced by

$$\Psi \to 1 \qquad (x^2 + z^2)^{1/2} \to \infty. \tag{1.18}$$

The resulting sequence of solitary waves was found to be qualitatively different from the three-dimensional sequence. Now, p and $\mathscr{E}$ (defined per unit y length) have no minima, but decrease monotonically with increasing U, becoming simultaneously zero as U reaches the speed of sound. Solutions in this limit obey the Kadomtzev–Petviashvili (1970) equation and can be written down in closed form (see Manakov *et al* 1977). At the other extreme ($U \to 0$) the solution represents a widely separated vortex pair mutually propelling each other in the z direction in obedience to Kelvin's theorem. The larger U, the smaller their separation and, for all sufficiently small p, vorticity is absent, as for the three-dimensional sequence. The solitary waves obtained by numerical integration are described in § 4. All our results are summarised in § 5.

2. Stretched dipole moment, impulse and energy for axisymmetric waves

Clearly $\Psi = 1$ obeys (1.14) and (1.15). Neighbouring states are obtained by writing

$$\Psi = 1 + \Psi' = 1 + \Psi'_r + i\Psi'_i \tag{2.1}$$

say, substituting into (1.14) and (1.15), linearising these with respect to Ψ', and separating their real and imaginary parts. In this way we obtain

$$2U\,\partial\Psi'_i/\partial z = -\nabla^2\Psi'_r + 2\Psi'_r \qquad 2U\,\partial\Psi'_r/\partial z = \nabla^2\Psi'_i \qquad (2.2), (2.3)$$

where

$$(\Psi'_r, \Psi'_i) \to 0 \qquad \text{for } r \equiv |\boldsymbol{x}| \to \infty. \tag{2.4}$$

Thus Ψ'_r and Ψ'_i obey

$$(\nabla^4 - 2\nabla^2 + 4U^2\,\partial^2/\partial z^2)(\Psi'_r, \Psi'_i) = 0. \tag{2.5}$$

We note that (2.5), though not (2.4), is satisfied by sinusoidal disturbances of wavenumber k, provided

$$U=\sqrt{\tfrac{1}{2}}(1+\tfrac{1}{2}k^2)^{1/2}. \tag{2.6}$$

This dispersion relationship for infinitesimal sound waves coincides with the usual Bogoliubov (1947) phonon spectrum. The solitary waves we seek below are all subsonic, i.e. $U<\sqrt{\tfrac{1}{2}}$.

The second use to which we put (2.1)–(2.5) is to determine the asymptotic form of axisymmetric wave solutions for $r\to\infty$, where the ∇^4 term of (2.5) is negligible to leading order. This is tantamount to discarding the $\nabla^2\Psi_r'$ term in (2.2) to leading order, so that

$$\Psi_r'=U\,\partial\Psi_i'/\partial z \qquad \nabla^2\Psi_i'=2U\,\partial\Psi_r'/\partial z \tag{2.7), (2.8}$$

and

$$(\nabla^2-2U^2\,\partial^2/\partial z^2)(\Psi_r',\Psi_i')=0. \tag{2.9}$$

The gradient terms in (1.11) play no part at these great distances; compressibility is the origin of the $2U^2\,\partial^2/\partial z^2$ term in (2.9). In the usual way we make the transformation

$$x'=x(1-2U^2)^{1/2} \qquad y'=y(1-2U^2)^{1/2} \qquad z'=z \tag{2.10}$$

which is real, since $U<\sqrt{\tfrac{1}{2}}$. This reduces (2.9) to Laplace's equation in (x',y',z') space. By (2.4) only the exterior harmonics are admissible, e.g. for axisymmetric disturbances we have, writing $\cos\theta'\equiv z'/r'$ and $r'\equiv|\boldsymbol{x}'|$,

$$\Psi_i'=-\sum_{n=0}^{\infty}m_n(r')^{-n-1}P_n(\cos\theta') \tag{2.11}$$

where $m_0, m_1, m_2, \ldots$ are the *stretched* monopole, dipole, quadrupole, ... coefficients, the adjective 'stretched' being added to distinguish these coefficients from the usually employed coefficients of $r^{-n-1}P_n(\cos\theta)$ in an expansion of Ψ_i' in inverse powers of r (an expansion which is here not useful since supplementary terms $r^{-n-1}P_{n+2}(\cos\theta)$, $r^{-n-1}P_{n+4}(\cos\theta), \ldots$ would be required by (2.9)).

We can obtain Ψ_r' from (2.11) by using (2.7). By symmetry, only odd n terms can appear in (2.11) for the solitary wave solution, and for sufficiently large r we may therefore write (replacing m_1 by m and setting $s=[(x^2+y^2)]^{1/2}$)

$$\Psi_r\sim 1+mU[2z^2-(1-2U^2)s^2][z^2+(1-2U^2)s^2]^{-5/2} \tag{2.12}$$

$$\Psi_i\sim -mz[z^2+(1-2U^2)s^2]^{-3/2} \tag{2.13}$$

for $r\to\infty$. For infinitely slow waves, such as the infinitely large vortex ring, m coincides with the customarily defined dipole moment of a ring vortex in an incompressible fluid. It was one of the objectives of the numerical work reported in § 3 to determine m for each solitary wave obtained. We note that, in the fluid description of § 1, (2.12) and (2.13) imply

$$\rho\sim 1+2mU[2z^2-(1-2U^2)s^2][z^2+(1-2U^2)s^2]^{-5/2} \tag{2.14}$$

$$S\sim 2\pi j-mz[z^2+(1-2U^2)s^2]^{-3/2} \tag{2.15}$$

$$\Phi\sim -\tfrac{1}{2}Us^2+m(1-2U^2)s^2[z^2+(1-2U^2)s^2]^{-3/2} \tag{2.16}$$

for $r\to\infty$. The following observation is important to what follows. While Ψ is necessarily single valued, S is not. The integer j in (2.15) is needed whenever (as for the vortex-type solutions) one or more closed circles of zero Ψ exist, since such circles of zero ρ essentially increase the connectivity of the fluid. But when no curves of zero ρ exist and the fluid occupies a simply connected region, we may write

$$j=0 \qquad S\sim\Psi_{\mathrm{i}} \qquad \text{for } r\to\infty. \tag{2.17), (2.18}$$

Such, for example, is the case when a solid body (without trailing vortices) moves through the fluid (see below).

We may use these asymptotic forms to prove that no non-trivial waves of permanent form exist that are completely stationary. If $\hat{\boldsymbol{x}}=\boldsymbol{x}/r$ is the unit radial vector, it follows from (1.8)–(1.13) that

$$\begin{aligned}\frac{\partial}{\partial x_j}&\{\hat{x}_i[\Sigma_{ij}-\rho u_i(u_j-U_j)]\}\\ &=\frac{1}{r}(\delta_{ij}-\hat{x}_i\hat{x}_j)[\Sigma_{ij}-\rho u_i(u_j-U_j)]\\ &=\frac{1}{r}[\tfrac{1}{2}(1-\rho)^2+(\hat{\boldsymbol{x}}\cdot\boldsymbol{\nabla}\rho^{1/2})^2+\rho(\hat{\boldsymbol{x}}\cdot\boldsymbol{u})\hat{\boldsymbol{x}}\cdot(\boldsymbol{u}-\boldsymbol{U})+\rho\boldsymbol{u}\cdot\boldsymbol{U}].\end{aligned} \tag{2.19}$$

When this is integrated over S_∞, the sphere at infinity, the surface integral on the left-hand side vanishes and we have

$$\int_{S_\infty}[\tfrac{1}{2}(1-\rho)^2+(\hat{\boldsymbol{x}}\cdot\boldsymbol{\nabla}\rho^{1/2})^2+\rho(\hat{\boldsymbol{x}}\cdot\boldsymbol{u})^2]\frac{\mathrm{d}V}{r}=\int_{S_\infty}[(\hat{\boldsymbol{x}}\cdot\boldsymbol{u})(\hat{\boldsymbol{x}}\cdot\boldsymbol{U})-\boldsymbol{u}\cdot\boldsymbol{U}]\frac{\rho\,\mathrm{d}V}{r}. \tag{2.20}$$

The left-hand side is positive definite, but the right-hand side vanishes when $U=0$. Thus for stationary solutions $\rho=1$ and $\hat{\boldsymbol{x}}\cdot\boldsymbol{u}=0$, and axisymmetry then requires that $\boldsymbol{u}=0$. This proof can be generalised to other potential energies of interaction, as for example when the potential energy $\frac{1}{2}(1-|\Psi|^2)^2$ in (1.14) is replaced by any $V(|\Psi|^2)$ that increases with $|1-|\Psi|^2|$ away from a minimum at $\Psi=1$.

By (2.14) the integral of $1-\rho$ over all space is non-zero, but the apparent violation of mass conservation is illusory, as we shall see below. This integral is also improper, i.e. its value depends on the *shape* of the volume V occupied by the fluid as the limit, $V\to V_\infty$, of infinite volumes is taken.

The occurrence of improper integrals in the theory of incompressible fluids is well known and understood. It arises there, as here, when we endeavour to evaluate the momentum (impulse) that should be assigned to a solitary wave such as a vortex. We can generalise this experience by adopting a well known way out of the difficulty (e.g. Batchelor 1967, § 7.2). In the classical argument, the impulse $\boldsymbol{p}$ is calculated in two steps, first the relatively simple one of integrating the momentum $\boldsymbol{p}_{\mathrm{I}}$ of the flow interior to $\mathscr{S}_0$, a very large sphere $r=\mathscr{R}$ ($\gg\bar{\omega}$), and second the conceptually more difficult one of assigning a momentum $\boldsymbol{p}_{\mathrm{E}}$ to the flow exterior to that sphere. In computing the latter it is noted that the flow far from the vortex coincides with that which would exist were $\mathscr{S}_0$ the real physical surface bounding a *solid* sphere $r=\mathscr{R}$. The force exerted on that sphere as the velocity of the sphere is increased from 0 to U is easily calculated, and when integrated provides the momentum supplied by external agencies to bring the sphere from rest to its final state of uniform motion U, i.e. it gives $\boldsymbol{p}_{\mathrm{E}}$ and hence $\boldsymbol{p}=\boldsymbol{p}_{\mathrm{I}}+\boldsymbol{p}_{\mathrm{E}}$.

The argument giving $\boldsymbol{p}_{\mathrm{E}}$ is hard to generalise for three reasons: first, it is evident from (2.16) that, though the spheroid

$$\mathscr{S}: \qquad s = \mathscr{R} \sin\theta \qquad z = (1-2U^2)^{1/2}\mathscr{R}\cos\theta \tag{2.21}$$

is for large $\mathscr{R}$ a streamline for one value of U, it is not a streamline for any other velocity of the body as it is accelerated from rest. Second, the stretched dipole moment

$$\bar{m} = \tfrac{1}{2}U\mathscr{R}^3(1-2U^2)^{1/2} \tag{2.22}$$

required to make $\mathscr{S}$ a streamline for velocity U, and the corresponding flow, are both so large for $\mathscr{R} \gg 1$ that the argument leading to (2.16) is invalidated. Third, healing of the wavefunction would generally occur on a solid body. If, for example, $\mathscr{S}$ were an infinite potential barrier to the bosons, Ψ and ρ would be zero on $\mathscr{S}$ and would rise to their bulk values in a healing layer of O(1) thickness surrounding $\mathscr{S}$. Such healing phenomena are of no interest for our present purposes. To overcome all these difficulties we consider *de novo* the problem of accelerating in the z direction from rest an axisymmetric body solid body B which for simplicity we suppose has an equatorial plane of symmetry, so that the flow it creates has the same symmetry (under $z \to -z$) as the solitary wave.

The force $\boldsymbol{F}$ required to accelerate B is given by

$$F_i = \int_{\mathscr{S}_{\mathrm{B}}} \Sigma_{ij}\,\mathrm{d}S_j \tag{2.23}$$

where $\mathrm{d}\boldsymbol{S}$ is the surface element drawn outwards from $\mathscr{S}_{\mathrm{B}}$, the surface of B. Using (1.8), (1.10) and the fact that

$$(\boldsymbol{u}-\boldsymbol{U})\cdot\mathrm{d}\boldsymbol{S} = 0 \qquad \text{on } \mathscr{S}_{\mathrm{B}} \tag{2.24}$$

where $\boldsymbol{U}$ is the velocity of B, we may rewrite (2.23) as

$$F_i = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{V}} \rho u_i\,\mathrm{d}V + \int_{\mathscr{S}} [\Sigma_{ij} - \rho u_i(u_j - U_j)]\,\mathrm{d}S_j \tag{2.25}$$

where $\mathscr{S}$ is a surface far from and surrounding $\mathscr{S}_{\mathrm{B}}$ but moving with it, $\mathscr{V}$ is the volume between $\mathscr{S}_{\mathrm{B}}$ and $\mathscr{S}$, and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{U}\cdot\boldsymbol{\nabla}$ is the derivative following the motion of B. We may rewrite (2.25) as

$$\boldsymbol{F} = -\mathrm{d}\boldsymbol{p}_{\mathrm{I}}/\mathrm{d}t + \boldsymbol{f} \tag{2.26}$$

where

$$f_i = \int_{\mathscr{S}} [\Sigma_{ij} - \rho u_i(u_j - U_j)]\,\mathrm{d}S_j \tag{2.27}$$

and $\boldsymbol{p}_{\mathrm{I}}$ is the momentum of the fluid interior to $\mathscr{V}$. Since $\rho - 1$ and $\boldsymbol{u}$ are $\mathrm{O}(r^{-3})$ for $r \to \infty$ we may, using (1.11), replace (2.27) by

$$f_i = -\int_{\mathscr{S}} [\tfrac{1}{2}(\rho-1)\delta_{ij} - \rho u_i U_j]\,\mathrm{d}S_j \tag{2.28}$$

and (1.9) now gives to the same accuracy

$$f_i = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{S}} S\,\mathrm{d}S_i + U_j\int_{\mathscr{S}} (u_i\,\mathrm{d}S_j - u_j\,\mathrm{d}S_i). \tag{2.29}$$

In this way, we obtain for the only case in which we are interested (axisymmetric $\mathscr{S}$ with f and U parallel to $0z$)

$$F = -\mathrm{d}p/\mathrm{d}t \qquad p = p_{\mathrm{I}} + p_{\mathrm{E}} \tag{2.30), (2.31}$$

where

$$p_{\mathrm{I}} = \int_{\mathscr{V}} \rho u_z \, \mathrm{d}V \qquad p_{\mathrm{E}} = -\int_{\mathscr{S}} S \, \mathrm{d}S_z \tag{2.32), (2.33}$$

and $\mathrm{d}S_z = \hat{z} \cdot \mathrm{d}\boldsymbol{S}$. It is important to note that, since vorticity cannot be created during the acceleration of B from rest, there are no curves of zero Ψ exterior to B, and we may use (2.17) and (2.18) to obtain

$$p_{\mathrm{E}} = -\int_{\mathscr{S}} \Psi_{\mathrm{i}} \, \mathrm{d}S_z. \tag{2.34}$$

Let us consider what happens if we accelerate B from rest to velocity U in a finite time τ, and thereafter maintain B at that velocity. The momentum transferred to the fluid can, at any instant t, be obtained by integrating (2.23) up to time t. Because of compressibility effects, that momentum will spread from B at a finite rate. Thus, even though the p of (2.31) will be unambiguously known at t, the division of p between p_{I} and p_{E} will depend on the speed of sound c, the distance $\mathscr{R}$ of $\mathscr{S}$ from B and so forth. If we wait a time long compared with $\mathscr{R}/c$ and τ, the flow in $\mathscr{S}$ will reach a steady state and at that time (2.32)–(2.34) will have attained their final steady-state values that can be obtained by solving (1.14) and (1.15) for constant U. The flow outside $\mathscr{S}$ will, in an unbounded fluid, never reach a steady state there being always†, at sufficiently great distances, transients that contradict the asymptotic laws (2.12)–(2.18). Nevertheless, (2.32)–(2.34) will be uniquely given by those laws and by writing

$$p_{\mathrm{I}} = \frac{1}{2\mathrm{i}} \int_{\mathscr{V}} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \, \mathrm{d}V \tag{2.35}$$

we obtain from (2.30) and (2.31)

$$p = \frac{1}{2\mathrm{i}} \int_{V_\infty} [(\Psi^* - 1) \nabla \Psi - (\Psi - 1) \nabla \Psi^*] \, \mathrm{d}V. \tag{2.36}$$

We have here appealed to the absolute convergence of the integral to replace $\mathscr{V}$ by V_∞, i.e. all space. Though (2.32)–(2.35) are all improper integrals, (2.36) is absolutely convergent.

In this paper we are more concerned with the momentum of a solitary wave than with that associated with the flow round a moving solid body. Since the asymptotic form of Ψ for $r \to \infty$ is the same in each case, integral (2.36) is unambiguous, and the contribution (2.35) made by p_{I} to p is clearly the momentum associated with the wave inside $\mathscr{V}$. Applicability of (2.36) is questionable only in its reliance on (2.34) for p_{E}. Against this two objections may be levelled, one trivial and one more profound. The trivial one is that solitary waves may possess vorticity and that, if so, S will not be

† If the fluid is of finite extent, sound waves reflected from the walls will eventually enter $\mathscr{S}$ again, and will carry the information about whether the container is free to move, or whether further momentum has been transferred at the walls to keep it at rest. After a sufficient number of repeated reflections, the fluid may reach a steady state. Asymptotic laws like (2.12)–(2.18) would then be valid with additional terms arising from the mass motion of the system. We shall not consider these complications.

single valued: reliance of (2.17) and (2.18) to pass from (2.33) to (2.34) seems misplaced. We may answer this by noting that in both cases (2.33) represents the momentum of the flow outside $\mathscr{V}$, and that (as $t\to\infty$ after the motion has been set up) this has the identical pseudo-dipolar form. Form (2.34) should therefore be true for both, and is preferable to (2.33) because it is unambiguous.

This answer touches on the more profound objection to (2.36). Irrespective of the outgoing waves outside $\mathscr{S}$, born during the acceleration of B, we know that (2.34) correctly gives the momentum outside $\mathscr{S}$ when a solid body is accelerated from rest. It is conceptually a much more difficult matter to visualise a sequence of operations that lead from an initial state of no motion to a final state of a solitary wave. Indeed, as we have seen above, no motionless solitary wave exists that could provide a starting point. It is now unclear whether (2.34) gives p_{E} correctly. For this reason we feel it is safer to call the p calculated from (2.36) not the momentum of the solitary wave but its 'impulse'. That (2.36) enjoys a special significance in the theory is corroborated by a variational argument below.

It is possible to relate the momentum p directly to the stretched dipole moment m, defined in (2.12) and (2.13). This was the preferred method used in § 3 to find m. If ω denotes solid angle and $\mathrm{d}S_s=\hat{\boldsymbol{s}}\cdot\mathrm{d}\boldsymbol{S}$

$$
\begin{aligned}
4\pi m &= m\int_{\mathscr{S}}\mathrm{d}\omega' = m\int_{\mathscr{S}}\mathrm{d}\left(\frac{z}{[z^2+(1-2U^2)s^2]^{1/2}}\right)\mathrm{d}\varphi\\
&=\int_{\mathscr{S}}\frac{m(1-2U^2)s\,\mathrm{d}S_s}{[z^2+(1-2U^2)s^2]^{3/2}}+(1-2U^2)\int_{\mathscr{S}}\frac{mz\,\mathrm{d}S_z}{[z^2+(1-2U^2)s^2]^{3/2}}\\
&=\int_{\mathscr{S}}\left(\mathrm{d}\boldsymbol{S}\times\frac{m(1-2U^2)s\hat{\boldsymbol{\varphi}}}{[z^2+(1-2U^2)s^2]^{3/2}}\right)_z-(1-2U^2)\int_{\mathscr{S}}\Psi_{\mathrm{i}}\,\mathrm{d}S_z\\
&=\int_{\mathscr{V}}\left[\boldsymbol{\nabla}\times\left(\frac{\Phi+\frac{1}{2}Us^2}{s}\right)\hat{\boldsymbol{\varphi}}\right]_z\mathrm{d}V-(1-2U^2)\int_{\mathscr{S}}\Psi_{\mathrm{i}}\,\mathrm{d}S_z\\
&=\int_{\mathscr{V}}[\rho u_z-U(1-\rho)]\,\mathrm{d}V-(1-2U^2)\int_{\mathscr{S}}\Psi_{\mathrm{i}}\,\mathrm{d}S_z\\
&=p+U\int_{\mathscr{V}}\left(1-|\Psi|^2+2U\frac{\partial\Psi_{\mathrm{i}}}{\partial z}\right)\mathrm{d}V\\
&=p+U\int_{\mathscr{V}}[1-|\Psi|^2-\nabla^2\Psi_{\mathrm{r}}-(1-|\Psi|^2)\Psi_{\mathrm{r}}]\,\mathrm{d}V. \qquad (2.37)
\end{aligned}
$$

We have here made use of (1.14), (1.17), (2.13), (2.16) and (2.36). In the limit $\mathscr{V}\to\infty$, (2.37) gives

$$
4\pi m=p+\tfrac{1}{2}U\int_{V_\infty}(1-|\Psi|^2)(2-\Psi-\Psi^*)\,\mathrm{d}V. \qquad (2.38)
$$

Again, the integral is properly convergent. Equation (2.38) confirms the classical value, $4\pi m$, for p in the large vortex ring limit ($U\to 0$), and also shows that at the other extreme ($U\to\sqrt{\frac{1}{2}}$) of the Tsuzuki-type solitary waves $p\sim\frac{8}{3}\pi m$; see appendix 1. Result (2.38) is also well confirmed by the numerical integrations of § 3.

The final quantity of interest is the energy $\mathscr{E}$ of the excitation. It is convenient to restore dimensional units temporarily. We form $\mathscr{E}$ by subtracting the energy of an undisturbed system of the same mass, for which $\Psi=\Psi_u=$ constant everywhere, from the energy of the system with a solitary wave, in which $\Psi\to\Psi_\infty=$ constant as $r\to\infty$. As Amit and Gross (1966) and Roberts and Grant (1971) have stressed, Ψ_∞ and Ψ_u differ slightly but significantly.

Equality of mass in the two systems requires

$$\int_V (\Psi_\infty^2-|\Psi|^2)\,\mathrm{d}V=(\Psi_\infty^2-\Psi_u^2)v \tag{2.39}$$

where v is the volume of V. It follows that

$$\begin{aligned}\mathscr{E}&\equiv\frac{\hbar^2}{2M}\int_V|\nabla\Psi|^2\,\mathrm{d}V+\frac{W_0}{2}\int_V|\Psi|^4\,\mathrm{d}V-\frac{W_0}{2}\Psi_u^4 v\\&=\frac{\hbar^2}{2M}\int_V|\nabla\Psi|^2\,\mathrm{d}V+\frac{W_0}{2}\int_V(\Psi_\infty^2-|\Psi|^2)^2\,\mathrm{d}V-\frac{W_0}{2}(\Psi_\infty^2-\Psi_u^2)^2 v.\end{aligned} \tag{2.40}$$

By (2.14), the integral on the left of (2.39), i.e. the integral of $1-\rho$ in dimensionless units, is improper, but in the limit $v\to\infty$ it is finite for all shapes of V. The fact that it is non-zero might raise qualms about our earlier arguments about impulse: returning to the accelerated body B, we recognise that asymptotic laws such as (2.14) in dimensional units give $\rho\to\rho_\infty=|\Psi_\infty|^2$, for $r\to\infty$ and not $\rho\to\rho_u=|\Psi_u|^2$, the undisturbed density. Although the difference is small, namely $O(1/v)$, it might seem that the body signals its change of motion to great distances faster than the speed of sound, in fact instantaneously. This is not true however. Ahead of the advancing sound waves set up by the acceleration of B, $\rho=\rho_u$, but behind that front ρ will tend to ρ_∞ for large (fixed) r; mass conservation is not violated. The same distinction is, it seems to us, appropriate for the solitary wave solutions.

Since the integral on the left of (2.39) is finite, the final term of (2.40) is $O(1/v)$ and will vanish in the limit $v\to\infty$. The remaining integrals are completely convergent. Reverting to dimensionless units, we have

$$\mathscr{E}=\tfrac{1}{2}\int_{V_\infty}|\nabla\Psi|^2\,\mathrm{d}V+\tfrac{1}{4}\int_{V_\infty}[1-|\Psi|^2]^2\,\mathrm{d}V. \tag{2.41}$$

Two points may be noted about (2.41). First, we may replace Ψ by $\Psi-1$ in the first integral of (2.41), integrate by parts (discarding the surface integral which vanishes by (2.12) and (2.13)), and appeal to (1.6) to obtain the general result

$$\mathscr{E}=Up+\tfrac{1}{4}\int_{V_\infty}(1-|\Psi|^2)|1-\Psi|^2\,\mathrm{d}V. \tag{2.42}$$

This was used in § 3 as a check on numerical accuracy. Second, we may perform the variation

$$\Psi\to\Psi+\delta\Psi \tag{2.43}$$

in the integrals (2.36) and (2.41). After discarding surface integrals that vanish provided $\delta\Psi\to 0$ for $r\to\infty$, we obtain

$$\delta p=\frac{1}{\mathrm{i}}\int_{V_\infty}\left[\delta\Psi^*\frac{\partial\Psi}{\partial z}-\delta\Psi\frac{\partial\Psi^*}{\partial z}\right]\mathrm{d}V \tag{2.44}$$

$$\delta\mathscr{E}=\tfrac{1}{2}\int_{V_\infty}\{\delta\Psi^*[-\nabla^2\Psi+(|\Psi|^2-1)\Psi]+\delta\Psi[-\nabla^2\Psi^*+(|\Psi|^2-1)\Psi^*]\}\,\mathrm{d}V. \tag{2.45}$$

Thus, we see that stationary values of $\mathscr{E}$, for fixed p and all $\delta\Psi$, require the Euler equation (1.14) and its complex conjugate to be satisfied, with U as Lagrange multiplier. Also we see that, if $\delta\Psi$ is the variation that takes one member of the sequence of solitary waves to the adjacent member, $\delta\mathscr{E}=U\delta p$. In other words

$$U=\partial E/\partial p \tag{2.46}$$

the derivative being understood to be taken along that sequence. Because of a non-vanishing surface integral in δp_{I}, this conclusion would not follow if instead of (2.36) we had used p_{I} for p, as given by (2.35). This gives added support to the argument that led to (2.36), for it is commonly held that (2.46) is a *sine qua non* for quasi-particle excitations such as the solitary waves. We may also note that (2.46) is obeyed by the sequences of classical vortex rings in an incompressible fluid (Roberts 1972, Norbury 1973). Equation (2.46) was well confirmed by the numerical results of §§ 3 and 4).

3. The numerical solutions for axisymmetric waves

We now address the problem of constructing axisymmetric solutions to the nonlinear system (1.14) subject to boundary condition (1.15). In view of the asymptotic expansions (2.12) and (2.13) at infinity, we introduce stretched variables r' and θ' based on (1.10). Also, since r' ranges from 0 to ∞, we introduce a new independent variable

$$\tilde{r}=r'/(R+r')$$

where R is constant. The domain of integration is now finite: $0<\tilde{r}<1$, $0<\theta'<\pi$. This more than compensates for the substantial complexity added to (1.14) by the transformation.

We expand Ψ_{r} and Ψ_{i} in double Chebyshev–Legendre series. Symmetry under the $\theta'\to\pi-\theta'$ transformation suggests that we take

$$\Psi_{\mathrm{r}}=\sum_{n=0}^{N}\sum_{m=0}^{M}a_{mn}T_m^*(\tilde{r})P_{2n}(\cos\theta') \tag{3.1}$$

$$\Psi_{\mathrm{i}}=\sum_{n=0}^{N}\sum_{m=0}^{M}b_{mn}T_m^*(\tilde{r})P_{2n+1}(\cos\theta') \tag{3.2}$$

where the $T_m^*(\tilde{r})$ are Chebyshev polynomials reduced to the [0, 1] interval: see for example Fox and Parker (1968). The boundary conditions at $\tilde{r}=0, 1$ and at $\theta'=0, \pi$ are ones of regularity that are already implicit in (3.1) and (3.2). No additional information need be incorporated at these boundaries. We are, however, required to evaluate a number of integrals, such as the energy integral (2.41), which now has the form

$$\mathscr{E}=\tfrac{1}{2}\int_{V_\infty}|\nabla\Psi|^2\,\mathrm{d}V+\pi\int_0^1\mathrm{d}\tilde{r}\int_0^{\frac{1}{2}\pi}(1-\Psi_{\mathrm{r}}^2-\Psi_{\mathrm{i}}^2)^2\frac{R^3\tilde{r}^2\sin\theta'\,\mathrm{d}\theta'}{(1-\tilde{r})^4(1-2U^2)^{1/2}}. \tag{3.3}$$

Provided the solutions obey (2.12) and (2.13), these integrals will converge at the

$\tilde{r}=1$ limit. For these asymptotic forms imply

$$\begin{aligned} &\Psi_r \to 1 \qquad \partial\Psi_r/\partial\tilde{r},\ \partial^2\Psi_r/\partial\tilde{r}^2,\ \Psi_i \text{ and } \partial\Psi_i/\partial\tilde{r} \to 0 \\ &\frac{\partial^3}{\partial\tilde{r}^3}\sum_{m=0}^{M} a_{m0}T_m^*(\tilde{r}) \to 0 \qquad \tilde{r}\to 1. \end{aligned} \tag{3.4}$$

Even if these conditions were not applied explicitly they would be satisfied approximately, but the errors arising from the finite truncation of the series in (3.1) and (3.2) would lead to difficulties. These are best avoided by imposing (3.4) directly on the expansions (3.1) and (3.2), so obtaining $5N+6$ relations between the coefficients a_{mn} and b_{mn}. We obtain a further $2N+2$ relations from the conditions

$$\partial\Psi_r/\partial\tilde{r} \qquad \Psi_i \to 0 \qquad \tilde{r}\to 0. \tag{3.5}$$

The remaining $2M(N+1)-(5N+6)$ equations are obtained by collocation at the zeros of the appropriate Chebyshev and Legendre polynomials. The total of $2(M+1)(N+1)$ nonlinear algebraic equations are solved by Newton–Raphson iteration.

The two input parameters are U, the speed of the solitary wave, and the scaling factor R. The integrals of the motion should be independent of R, and this was found to be the case provided R was representative of the length scale over which the solution varies. If R is much greater than or much less than this, the number of polynomials required for accurate answers becomes large. For $0.5<U<0.65$, any value of R in the range $2<R<4$ gave the same results to three figure accuracy. At $U=0.69$, R in the range $5<R<7$ gave the same results to that accuracy but, for $U\leqslant 0.4$, N needs to be larger to obtain results correct to three figures.

The Newton–Raphson iteration procedure requires an initial approximation for the solution at the U selected. This was provided by the asymptotic theory for the large circular vortex given by Grant and Roberts (1974), which is valid for $U\to 0$. Once an accurate numerical solution had been found for small U, U was slowly increased until the complete family, whose properties are summarised in table 1, had been obtained. The integrals were evaluated by Gaussian quadrature.

In table 1 we show $\mathscr{E}$ as calculated from (2.41), p from (2.42) and m from (2.38). The value of p was checked by computing (2.32) and (2.33), and m was compared with the estimate obtained by directly evaluating

$$\frac{\partial^2}{\partial\tilde{r}^2}\sum_{m=0}^{M} b_{m0}T_m^*(1)$$

Table 1.

U	$\mathscr{E}$	p	m	$\tilde{\omega}$
0.4	129.0	233.0	22.6	3.36
0.5	80.7	123.5	13.0	2.31
0.55	66.5	96.5	10.6	1.82
0.60	56.4	78.9	8.97	1.06
0.63	52.3	72.2	8.37	—
0.66	50.7	69.6	8.20	—
0.68	53.7	74.1	8.80	—
0.69	60.0	83.2	9.92	—

and applying (2.13). A further check was made by differencing the results of table 1 to obtain $\Delta E/\Delta p$ and comparing this with U; see (2.46). Yet another test was to compare results at small U with the asymptotic theory of Grant and Roberts (1974). All these checks on accuracy were satisfactorily survived by the numerical solution.

As a further independent check, a completely different programme was constructed that used a finite difference method to solve (1.14) and (1.15). Stretched cylindrical polar coordinates were introduced, related to s and z by

$$\hat{s} = s/(R+s) \qquad \hat{z} = z/(R+z). \tag{3.6}$$

The transformed (1.14) was expressed in second-order finite difference form, and the resulting nonlinear equations were solved by Newton–Raphson iteration, using a sparse matrix inversion routine. The agreement of the results with those obtained by the first method was extremely satisfactory. One disadvantage of the finite difference method is that it is almost impossible to obtain a direct estimate of the stretched dipole moment from the behaviour of the solution near $\hat{s}=1$, $\hat{z}=1$. The corresponding step in the series expansion method was simple to take.

The stream function Φ defined by (1.17) was also obtained. This function measures, in the frame co-moving with the vortex, the flux of fluid through the surface of revolution formed by rotating the streamline concerned about the z axis: naturally, $\Phi=0$ on $s=0$, and Φ on other streamlines can be found by integrating along $\hat{r}$ constant.

Contours of ρ and Φ are shown in figure 1 in four cases: $U=0.3$, 0.55, 0.66 and 0.69. In each case the box size is 30×18 non-dimensional units. As U increases from 0 to approximately 0.62, the dimension of the 'vortex', as judged by the closed $\Phi=0$ streamline, diminishes monotonically. At $U\simeq 0.62$ it disappears entirely, and the velocity potential is thereafter single valued, without the necessity for branch cuts. The energy, impulse and stretched dipole moment up to now decreased steadily. This soon ceased, however. When $U\simeq 0.657$, $\mathscr{E}$ and p simultaneously attained minima of approximately 50.7 and 69.6; here $m\simeq 8.20$. As U increased beyond 0.657 to $\sqrt{\frac{1}{2}}$, the dimensions of the wave began to increase, the s scale doing so rather more rapidly than the z scale. Simultaneously the density approached unity everywhere. The scale increases were so large, however, that $\mathscr{E}$ and p increased without limit as $U\to\sqrt{\frac{1}{2}}$. It is interesting to note that in all cases regions of positive density excess $\rho-1$ exist, although such excesses disappear both as $U\to 0$ and $U\to\sqrt{\frac{1}{2}}$.

As can be seen from table 1, $\mathscr{E}$ and p have smooth minima as functions of U. When these results are plotted in the $p\mathscr{E}$ plane however, there is a cusp at the minima; see figure 2. The existence of a cusp, rather than a smooth minimum, is a consequence of the established non-existence of solutions with $U=\partial E/\partial p=0$; see (2.20) and (2.46). Also shown in table 1 is the radius, $\hat{\omega}$, of the zero $|\Psi|$ circle in the solution (when such a curve exists).

Although (see § 1) the condensate cannot be expected to model helium with quantitative accuracy, it is of some interest to place our results on the dispersion diagram for superfluid ^{4}He. To do this, we must re-interpret our results in dimensional form. There is no doubt that we should scale so that the unit of circulation h/M is 9.967×10^{-4} cm^2 s^{-1} and the bulk density is about 0.145 g cm^{-3} (Woods 1972). Our third choice is more contentious. We can either make the healing length agree with (for example) the value of 1.13 Å given by Padmore (1972), or we can choose the speed of sound to conform with the experimental value, e.g. 2.3×10^4 cm s^{-1} as given by Chase (1953). The latter course would imply a healing length of only 0.49 Å.

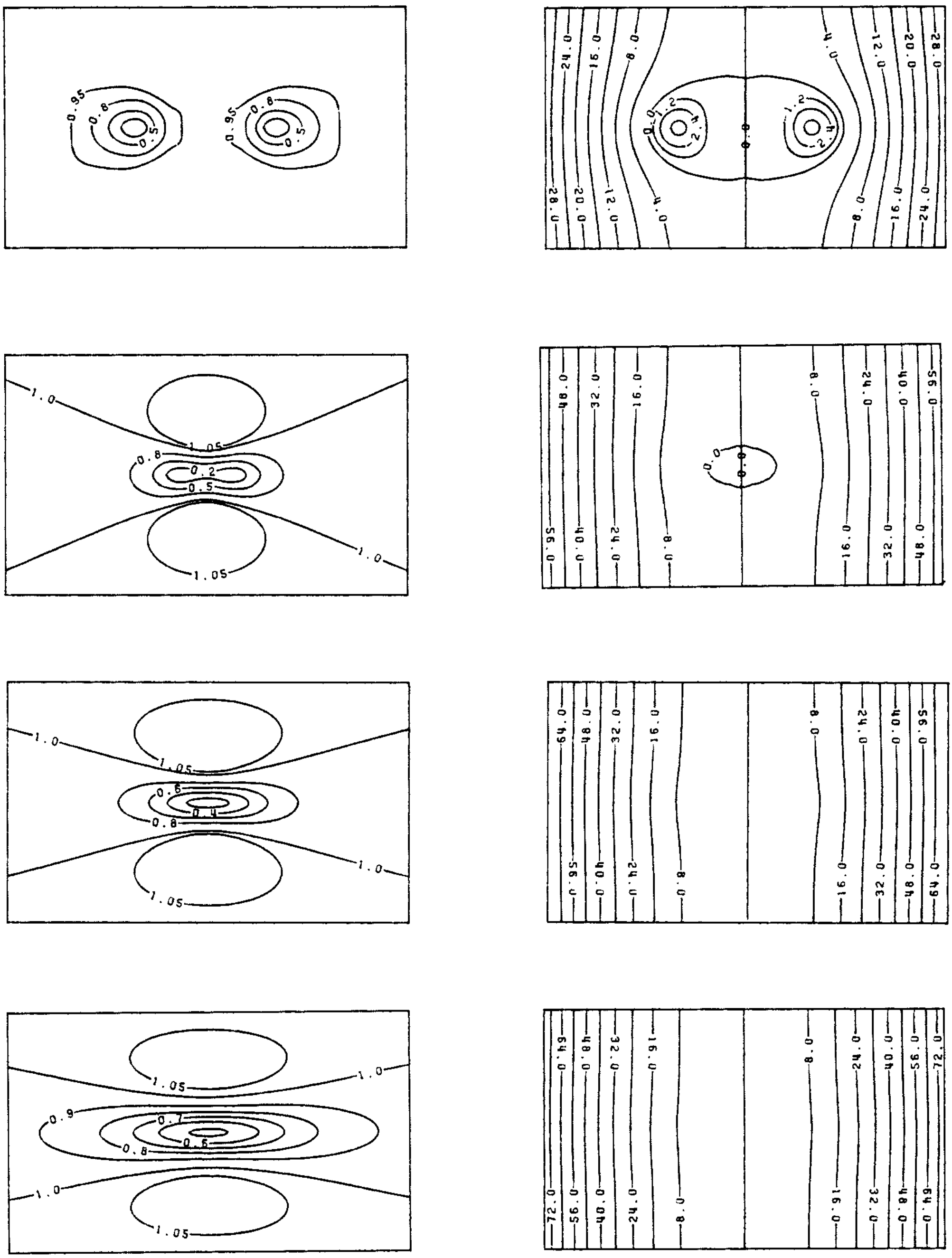

Figure 1. Equidensity surfaces (left) and streamlines (right) for four axisymmetric solitary waves, moving with dimensionless velocities $U = 0.3$ (top), 0.55, 0.66 and 0.69 (bottom). The contours are marked with density ρ (left) and with the value of the streamfunction Φ (right), both in the dimensionless units defined in § 1.

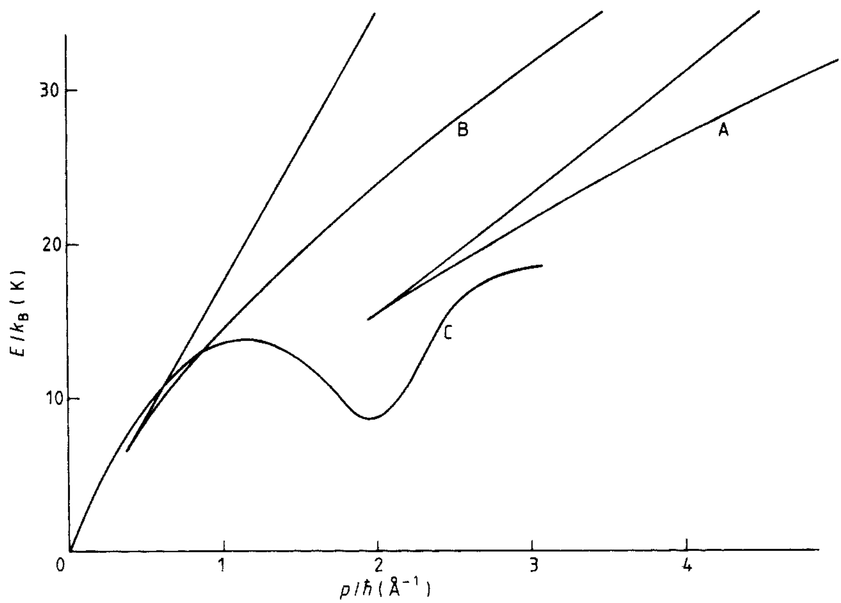

Figure 2. Showing the momentum–energy curve of the axisymmetric solitary wave solutions. Dimensional units are based on a density ρ of 0.145 g cm^{-3}, and a quantum of circulation $\kappa = h/M$ of 0.9967×10^{-3} cm^2 s^{-1}. On curve A, the healing length was taken to be 1.13 Å; on curve B, the same results are given based instead on the sound speed of 230 m s^{-1}. The observed dispersion curve C is shown for comparison purposes: for this $c = 238$ m s^{-1}.

Since it was not clear to us which alternative was to be preferred, we give both curves in the $p\mathscr{E}$ diagram of figure 2, on which the excitation spectrum obtained by Woods (1972) is also shown. In neither case is the cusp far from the experimental curve.

4. Two-dimensional solutions

Particularly in view of interest in thin helium films, we examine briefly here the analogous solutions in two dimensions, namely the family of solitary waves in which $\Psi = \Psi(x, z - Ut)$ and which also starts at $U = 0$ with the widely separated vortex pair, corresponding to zeros in Ψ at $(0, \pm\bar{\omega})$.

The analysis proceeds much as before. (1.17) is replaced by

$$\boldsymbol{u} = U\hat{\boldsymbol{z}} + \rho^{-1}\boldsymbol{\nabla}\times(\Phi\hat{\boldsymbol{y}}) \tag{4.1}$$

and the asymptotic forms corresponding to (2.12), (2.13), (2.16) are

$$\Psi_i \sim -mz[z^2 + (1-2U^2)x^2]^{-1} \tag{4.2}$$

$$\Psi_r \sim 1 + m\{U[z^2 - (1-2U^2)x^2] - \tfrac{1}{2}mz^2\}[z^2 + (1-2U^2)x^2]^{-2} \tag{4.3}$$

$$\Phi \sim -Ux + m(1-2U^2)x[z^2 + (1-2U^2)x^2]^{-1} \tag{4.4}$$

for $(x^2+z^2)^{1/2} \to \infty$. The proof that $U = 0$ solutions do not exist follows from, and relies on, (4.2) and (4.3). (Evidently $U = 0$ for an isolated vortex line, for which a monopole term replaces (4.2).) The derivations of (2.36) and (2.41) are changed only in that $\mathrm{d}V$ is replaced by $\mathrm{d}x\,\mathrm{d}z$, and provide respectively the impulse and energy per unit y length of the solitary wave. With that interpretation (2.42) and (2.46) are unchanged. The argument leading to (2.38) is readily modified to give

$$2\pi m(1-2U^2)^{-1/2}=p+\tfrac{1}{2}U\int_{V_\infty}(1-|\Psi|^2)(2-\Psi-\Psi^*)\,\mathrm{d}x\,\mathrm{d}z. \tag{4.5}$$

The second of the two numerical methods described in § 3, the finite difference method with stretched coordinates $\hat{x}$ and $\hat{z}$ defined as in (3.6), was used to construct table 2. The quantities $\mathscr{E}$, p and m were computed by numerical evaluation of the two-dimensional equivalents of (2.41) and (2.42), together with (4.5). As for the circular ring, $\tilde{\omega}$ decreases to zero as U is increased, and subsequently vorticity is absent. There is, however, a striking difference thereafter from the axisymmetric solutions. The impulse and energy per unit length have no minima, and as $U\to\sqrt{\tfrac{1}{2}}$, the speed of sound, both p and $\mathscr{E}$ tend to zero, with $\mathscr{E}\sim p/\sqrt{2}$. Indeed the excitations have slightly lower energy/momentum than the phonons. For the phonons follow the Bogoliubov spectrum (2.6) so that

$$\frac{\mathscr{E}}{p}\equiv U\sim\frac{1}{\sqrt{2}}+\frac{1}{8\sqrt{2}}k^2+\ldots \qquad k\to 0 \tag{4.6}$$

i.e. $\mathscr{E}/p$ exceeds $1/\sqrt{2}$, while for our solitary waves we have

$$\frac{\mathscr{E}}{p}\equiv U\sim\frac{1}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\left(\frac{3}{8\pi\sqrt{2}}\right)^2p^2 \qquad p\to 0 \tag{4.7}$$

i.e. $\mathscr{E}/p$ is less than $1/\sqrt{2}$. There seems to be no obvious reason why these states should not be thermally populated to a slightly higher density than the phonons and make a slightly greater contribution to the low-temperature specific heat. (We are grateful to Professor S Putterman for this observation.)

Table 2.

U	$\mathscr{E}$	p	m	$\tilde{\omega}$
0.3	10.0	19.6	4.19	1.75
0.4	8.16	14.1	3.55	0.89
0.5	6.40	10.2	3.16	—
0.6	4.49	6.71	2.91	—
0.65	3.28	4.77	2.84	—
0.68	2.29	3.28	2.83	—
0.69	1.83	2.60	2.83	—

Results such as (4.7) are derived from the asymptotic theory developed in the appendix, which capitalises on the great increase in scale of the wave in the limit $U\to 1/\sqrt{2}$, which occurs for the two-dimensional solutions as it did for the axisymmetric solutions. Unlike the latter case however, the governing equation can be solved in closed form. In this way we obtain (4.7) and also find that $m\to 2\sqrt{2}$ for $U\to 1/\sqrt{2}$. In the opposite limit $U\to 0$ of the widely separated vortex pair, $m\to p/2\pi$. The complete $p\mathscr{E}$ curve is shown in non-dimensional units in figure 3.

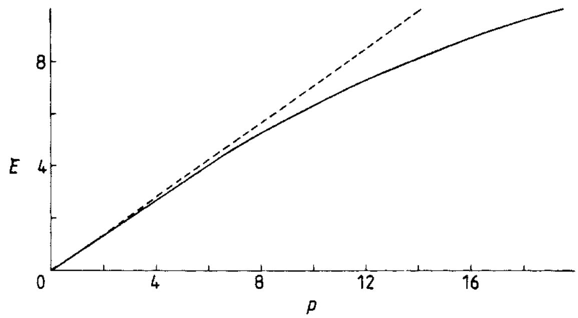

Figure 3. Showing the momentum–energy curve of the two-dimensional (xz) solitary wave solutions, using the dimensionless units defined in § 1. Momentum and energy here refer to unit y length. The broken line gives the speed of sound $(E = p/\sqrt{2})$.

5. Conclusions

A complete family of axisymmetric solutions for the nonlinear Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M} \nabla^2 \Psi + W_0 |\Psi|^2 \Psi - ME_v \Psi \tag{5.1}$$

has been derived numerically. These are waves of permanent form moving with a velocity U which has also been determined numerically, i.e.

$$\Psi(x, y, z, t) = \Psi(x, y, z - Ut) \tag{5.2}$$

where $0z$ is the axis of symmetry. These waves are solitary since

$$\Psi \to (ME_v/W_0)^{1/2} = \rho_\infty^{1/2} \text{ (say)} \qquad \text{as } r \to \infty \tag{5.3}$$

i.e. at great distances the solution approaches the undisturbed state $\Psi = \rho_\infty^{1/2}$, which is an exact solution of (5.1).

The solutions are related to superfluidity, as modelled by the imperfect Bose condensate, in which the single-particle wavefunction Ψ obeys (5.1), with M the mass of the boson, W_0 the strength of a δ-function repulsive potential between bosons, E_v is the energy level per unit mass of a boson in the undisturbed state in which the mass density is ρ_∞ everywhere. Expressions for the momentum (or impulse) p and energy $\mathscr{E}$ of the excitations were obtained, and it was demonstrated that their velocity is the derivative $\partial \mathscr{E}/\partial p$, taken along the sequence. Since (as was shown) U cannot be zero, p and $\mathscr{E}$ must achieve their minima simultaneously. This was confirmed numerically, and the corresponding cusp was located in the $p\mathscr{E}$ plane. In that plane the sequence consists of two branches meeting at the cusp. For each momentum greater than its minimum, $p_{\min}$, at the cusp, two states of different energy are possible. The lower branch (for all $p > p_0$, say approximately 0.62) consists of vortices. The smaller the p the smaller their radius $\tilde{\omega}$, the faster they move, and the smaller their spatial extent. For $p = p_0$, $\tilde{\omega} = 0$. For $p_{\min} < p < p_0$ vorticity is absent, and the disturbance resembles a rarefaction pulse. As p increases along the upper branch, the spatial extent of the pulse increases in all directions, particularly in directions perpendicular to the symmetry axis; at the same time the amplitude of the disturbance becomes smaller, but not sufficiently rapidly to prevent p and $\mathscr{E}$ from increasing indefinitely. As $p \to \infty$ along the upper branch $U \to c$, the speed of sound $E_v^{1/2}$. The disturbance is then almost one dimensional, i.e. dependent on z alone. The very existence of p_0 may have interesting implications for nucleation theory since, as p crosses p_0, a quantum of circulation is smoothly created or destroyed.

One of the objectives of the present paper was to clarify Onsager's concept of the roton as 'the ghost of a vanished vortex ring'. It was hoped to show that as U increased from zero, the large vortex ring would become more localised, vorticity would disappear, and the resulting sound pulse would merge, in the $p\mathscr{E}$ plane, with the phonon branch of the dispersion curve. Although in the shrinkage of the disturbance and the vanishing of $\tilde{\omega}$, this expectation has been fulfilled, the subsequent increase of p and $\mathscr{E}$, and indeed the very existence of the upper branch, came as a surprise. The existence of a minimum of p and $\mathscr{E}$ seems to rule out merging of the sequence with the phonon branch.

It seems to us that the idea of the roton as a ghostly vortex ring could best be saved in one of two ways. First, it might be demonstrated that the Bose condensate is too crude a model of helium to be reliable, and that in a better model our sequence would have no minimum momentum (other than zero). Second it might be shown that, if our sequence intersects the phonon–roton dispersion curve (as in curves B and C on figure 2), level crossing phenomena would reconnect the curves so that our lower branch joined smoothly to the observed dispersion curve (Turkevich 1981). In this context it is interesting to note that our lower branch always (i.e. no matter what values of a or c we choose; see § 3) intersects the line $\mathscr{E} = cp$. To take these speculations further is clearly beyond the scope of this paper.

In addition to these axisymmetric solutions, a complete family of two-dimensional solitary waves was located in § 4. It is hoped that this will be of at least qualitative usefulness to those interested in helium films. The family shares similarities with the axisymmetric sequence, e.g. vorticity is absent for all sufficiently large U. At this end of the sequence, the solution represents an oppositely directed pair of parallel line vortices, whose separation $2\tilde{\omega}$ ($\sim \kappa/2\pi U$, for $U \to 0$) becomes infinite as $U \to 0$. The striking difference from the axisymmetric family occurs at the other end of the sequence. In two dimensions p and $\mathscr{E}$ (now defined per unit length) have no minima, and vanish together as U approaches the speed of sound. The sequence thus merges tangentially with the phonon branch of the dispersion curve in the $p\mathscr{E}$ plane.

Acknowledgments

We are grateful to Professor S J Putterman for his interest in this work. He and one of us (PHR) wish to thank the Science and Engineering Research Council for support under Grant GR/B64215. We are grateful also to our colleague, Dr R S Johnson, for bringing the paper by Manakov *et al* (1977) to our notice.

Appendix 1. The $p \to \infty$ limit on the upper branch

On substituting

$$\Psi = f + \mathrm{i}g \tag{A1}$$

into (1.14) and equating real and imaginary parts, we obtain

$$2U\,\partial g/\partial z = -\nabla^2 f + f(f^2+g^2-1) \tag{A2}$$

$$-2U\,\partial f/\partial z = -\nabla^2 g + g(f^2+g^2-1). \tag{A3}$$

Now seek solutions of the form

$$f = 1+\varepsilon^2 f_1 + \varepsilon^4 f_2 + \ldots \tag{A4}$$

$$g = \varepsilon g_1 + \varepsilon^3 g_2 + \ldots \tag{A5}$$

$$U = U_0 + \varepsilon^2 U_1 + \varepsilon^4 U_2 + \ldots \tag{A6}$$

and stretch the independent variables by writing

$$\xi = \varepsilon^2 x \qquad \eta = \varepsilon^2 y \qquad \zeta = \varepsilon z. \tag{A7}$$

On substitution into (A2) and (A3) the leading order parts are found to be $O(\varepsilon^2)$ and $O(\varepsilon^3)$ respectively, and give

$$2U_0 \frac{\partial g_1}{\partial \zeta} = 2f_1 + g_1^2 \tag{A2$_1$}$$

$$-2U_0 \frac{\partial f_1}{\partial \zeta} = -\frac{\partial^2 g_1}{\partial \zeta^2} + (2f_1 + g_1^2)g_1 \tag{A3$_1$}$$

whence

$$U_0 = \sqrt{\tfrac{1}{2}} \qquad 2f_1 + g_1^2 = \sqrt{2}\,\partial g_1/\partial \zeta. \tag{A8), (A9}$$

To leading order, the impulse (2.36) of the wave is by (A9) (with $\sigma = \varepsilon^2 s$)

$$\begin{aligned} p &= \frac{2\pi}{\varepsilon}\int \left(f_1 \frac{\partial g_1}{\partial \zeta} - g_1 \frac{\partial f_1}{\partial \zeta}\right)\sigma\,\mathrm{d}\sigma\,\mathrm{d}\zeta = \frac{4\pi}{\varepsilon}\int f_1 \frac{\partial g_1}{\partial \zeta}\,\sigma\,\mathrm{d}\sigma\,\mathrm{d}\zeta \\ &= \frac{4\pi}{\varepsilon}\int \left[\frac{1}{\sqrt{2}}\left(\frac{\partial g_1}{\partial \zeta}\right)^2 - \frac{1}{2} g_1^2 \frac{\partial g_1}{\partial \zeta}\right]\sigma\,\mathrm{d}\sigma\,\mathrm{d}\zeta = \frac{2\pi\sqrt{2}}{\varepsilon}\int \left(\frac{\partial g_1}{\partial \zeta}\right)^2 \sigma\,\mathrm{d}\sigma\,\mathrm{d}\zeta. \end{aligned} \tag{A10}$$

To the same order (2.42) shows that

$$\mathscr{E} = U_0 p \tag{A11}$$

and by (2.38) and (A9)

$$4\pi m = p + U_0 \frac{2\pi}{\varepsilon}\int \left(\frac{\partial g_1}{\partial \zeta}\right)^2 \sigma\,\mathrm{d}\sigma\,\mathrm{d}\zeta = \frac{3p}{2}$$

by (A10). Thus, consistent with the result of § 3,

$$p = \tfrac{8}{3}\pi m. \tag{A12}$$

To this order f_1 and g_1 are undetermined; we know only that they are related by (A9). More information is obtained at the next order, namely, $O(\varepsilon^4)$ in (A2) and $O(\varepsilon^5)$ in (A3). These give

$$\sqrt{2}\frac{\partial g_2}{\partial \zeta} + 2U_1 \frac{\partial g_1}{\partial \zeta} = -\frac{\partial^2 f_1}{\partial \zeta^2} + (2f_2 + 2g_1 g_2 + f_1^2) + f_1(2f_1 + g_1^2) \tag{A2$_2$}$$

$$-\sqrt{2}\frac{\partial f_2}{\partial \zeta} - 2U_1 \frac{\partial f_1}{\partial \zeta} = -\frac{\partial^2 g_2}{\partial \zeta^2} - \nabla_H^2 g_1 + g_2(2f_1 + g_1^2) + g_1(2f_2 + 2g_1 g_2 + f_1^2) \tag{A3$_2$}$$

where $\nabla_H^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$. If we define

$$\rho_2 = 2f_2 + 2g_1g_2 + f_1^2 = 2f_2 + 2g_1g_2 + \frac{1}{2}\left(\frac{\partial g_1}{\partial\zeta}\right)^2 - \frac{1}{\sqrt{2}} g_1 \frac{\partial g_1}{\partial\zeta} + \frac{1}{4} g_1^4 \quad \text{(A13)}$$

$$S_2 = \frac{\partial}{\partial\zeta}(g_2 - f_1g_1 - \tfrac{1}{3}g_1^3) = \frac{\partial g_2}{\partial\zeta} - \frac{g_1}{\sqrt{2}}\frac{\partial^2 g_1}{\partial\zeta^2} - \frac{1}{\sqrt{2}}\left(\frac{\partial g_1}{\partial\zeta}\right)^2 + \frac{1}{2} g_1^2 \frac{\partial g_1}{\partial\zeta} \quad \text{(A14)}$$

we see from $(\text{A}2)_2$ and $(\text{A}3)_2$ that

$$\rho_2 - \sqrt{2}\, S_2 = 2U_1 \frac{\partial g_1}{\partial\zeta} + \frac{1}{\sqrt{2}}\frac{\partial^3 g_1}{\partial\zeta^3} - \left(\frac{\partial g_1}{\partial\zeta}\right)^2 \quad \text{(A15)}$$

$$\frac{\partial}{\partial\zeta}(\rho_2 - \sqrt{2}\, S_2) = -2U_1 \frac{\partial^2 g_1}{\partial\zeta^2} + \sqrt{2}\,\nabla_H^2 g_1 + 4\frac{\partial g_1}{\partial\zeta}\frac{\partial^2 g_1}{\partial\zeta^2}. \quad \text{(A16)}$$

Consistency of these equations demands that

$$2\sqrt{2}\, U_1 \frac{\partial^2 g_1}{\partial\zeta^2} - \nabla_H^2 g_1 + \frac{\partial}{\partial\zeta}\left[\frac{1}{2}\frac{\partial^3 g_1}{\partial\zeta^3} - \frac{3}{\sqrt{2}}\left(\frac{\partial g_1}{\partial\zeta}\right)^2\right] = 0. \quad \text{(A17)}$$

The one-dimensional form ($\nabla_H^2 = 0$) of this equation has been obtained and integrated by Tsuzuki (1971) to give a form of the Korteweg–de Vries equation. The axisymmetric solutions of (A17), in which $g_1 = g_1(\sigma, \zeta)$ where $\sigma = \varepsilon^2 s$, appear (for $\varepsilon \to 0$) to provide the asymptotic $p \to \infty$ form of the solution on the upper branch of the waves determined numerically in § 3.

The two-dimensional form of (A17) governing $g_1 = g_1(\xi, \zeta)$ is the Kadomtsev–Petviashvili (1970) equation to which simple solutions have been obtained by Manakov *et al* (1977). The particular solution relevant to the two-dimensional solitary waves discussed in § 4 is

$$g_1 = -2\sqrt{2}\zeta/(\xi^2 + \zeta^2 + \tfrac{3}{2}) \qquad U_1 = -1/2\sqrt{2} \quad \text{(A18), (A19)}$$

so that

$$f_1 = -2/(\xi^2 + \zeta^2 + \tfrac{3}{2}). \quad \text{(A20)}$$

The scaling (A7) for x and z (there is no dependence on y so the second of (A7) is irrelevant) and the expressions (2.36) and (2.41) (appropriately modified by replacing $\mathrm{d}V$ by $\mathrm{d}x\,\mathrm{d}z$) show that the momentum and energy per unit length are proportional to ε. In fact (A11) and (A18) give

$$\mathscr{E} = \tfrac{8}{3}\pi\varepsilon = p/\sqrt{2}. \quad \text{(A21)}$$

This is consistent with the numerical results of § 4.

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